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POLYNOMIAL INVARIANTS OF 2-BRIDGE KNOTS THROUGH 22 CROSSINGS

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ABSTRACT. We calculate the homfly, Kauffman, Jones, Q, and Conway polynomials of 2-bridge knots through 22 crossings and list all the pairs sharing the same polynomial invariants.

1. INTRODUCTION

A simple question for polynomial invariants of knots is: "How many knots do they classify?" Concerning this problem, we made a computer experiment with 2-bridge knots, which are completely classified by Schubert [25]. We calculated the homfly, Kauffman, Jones, Q, and Conway polynomials of 2-bridge knots through 22 crossings, and searched all the pairs of 2-bridge knots having the same polynomial invariants. The total number of the knots is 350,207, where each chiral pair is counted as one knot. If a chiral pair is counted separately, then this amounts to 699,732. The program is written in Turbo Pascal for the NEC PC-9801 Series. In a sequel to this paper, we shall report on 2-bridge links.

The homfly polynomial $P_L \in Z[v^{\pm 1}, z^{\pm 1}]$ [6, 23] of an oriented link L is defined, as in [20], so that

$$v^{-1}P_{L_+} - vP_{L_-} = zP_{L_0},$$

where (L_+, L_-, L_0) is a skein triple. Putting $v = 1$, we get the Conway polynomial $\nabla_L \in Z[z]$ [3] (substituting $(v, z) = (1, t^{1/2} - t^{-1/2})$, we get the Alexander polynomial), and substituting $(v, z) = (t, t^{1/2} - t^{-1/2})$, we get the Jones polynomial $V_L \in Z[t^{\pm 1/2}]$ [8]. These are skein invariants. We refer to [18] for the definitions of skein triple and skein equivalence. The Kauffman polynomial $F_L \in Z[a^{\pm 1}, z^{\pm 1}]$ of an oriented link L is given by $F_L = a^{-w}\Lambda_D$, where Λ_D is the L-polynomial of a diagram D of L and w is the writhe of D. We refer to [13] for the definitions of a writhe and the L-polynomial. Putting $a = 1$, we get the Q polynomial $Q_L \in Z[z^{\pm 1}]$ [1, 7] of an unoriented link $|L|$, and substituting $(a, z) = (-t^{-3/4}, t^{-1/4} + t^{1/4})$, we get the Jones polynomial [16].

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If L is a 2-bridge knot or link, then we have

(1) $$Q_L(z) = 2z^{-1}V_L(t)V_L(t^{-1}) + 1 - 2z^{-1},$$

where $z = -t - t^{-1}$ [11]. Thus, if we know the Jones polynomial of a 2-bridge knot or link, we can deduce the Q polynomial. In an early computer calculation of the polynomial invariants of 2-bridge knots and links, we found many pairs of 2-bridge knots and links with the same Q polynomial but distinct Jones polynomial, except for a reflection such as right- and left-handed trefoils. This has been generalized to the following theorem in [12]:

For any positive integer N, there exist N sets of 2^N 2-bridge knots S_1, $S_2, \dots, S_N$ with $S_i = \{K_{i1}, K_{i2}, \dots, K_{i\,2^N}\}$ such that: all the knots in $\bigcup_{i=1}^N S_i$ have the same Q and Conway polynomials; all the knots in each S_i are skein equivalent; and all the knots $K_{11}, K_{21}, \dots, K_{N1}$ have mutually distinct Jones polynomials.

In addition, we observe the following for 2-bridge knots through 22 crossings:

Fact 1. *$P_K(v, z) = P_{K'}(v, z)$ if and only if $V_K(t) = V_{K'}(t)$ and $\nabla_K(z) = \nabla_{K'}(z)$.*

Fact 2. *K is amphichiral if and only if $V_K(t) = V_K(t^{-1})$ (or $P_K(v, z) = P_K(v^{-1}, z)$).*

Fact 3. *The number of knots having the same homfly or Kauffman polynomial is at most two.*

Regarding Fact 3, we can construct the following examples:

(i) Arbitrarily many 2-bridge knots with the same Jones polynomial ([9, Theorem 6]).

(ii) Arbitrarily many fibred, amphichiral, skein equivalent 2-bridge knots ([10, Theorem 1]).

(iii) A pair of fibred, amphichiral, skein equivalent 2-bridge knots with the same Kauffman polynomial ([10, Theorem 4]).

(iv) A pair of 2-bridge knots with the same Kauffman polynomial but distinct Alexander polynomials ([10, Theorem 5]).

Note that (ii) above does not necessarily include (i) because the 2-bridge knots constructed in (i) may have distinct Conway polynomials. For the Kauffman polynomial we shall give an example similar to (i) in a forthcoming paper.

2. Formulas

Let S_1 and S_2 be the elementary braids generating the 3-braid group as shown in Figure 1.

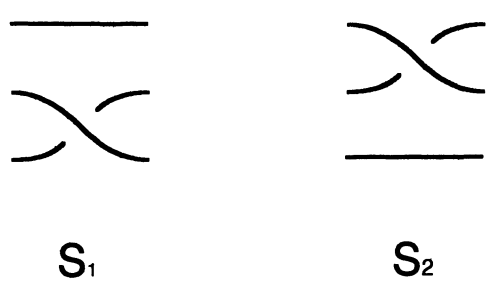

FIGURE 1

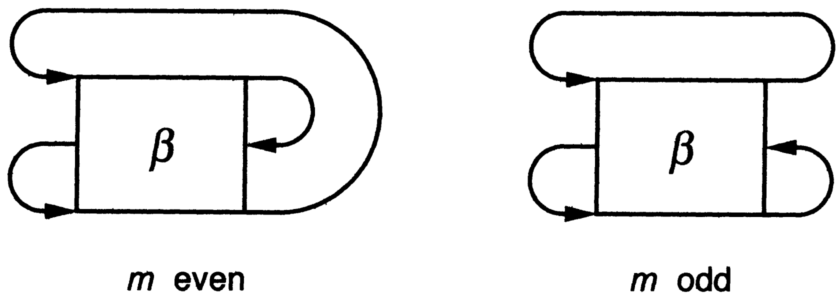

FIGURE 2

Let $D(b_1, b_2, \ldots, b_m)$ be the oriented 2-bridge knot (m is even) or link (m is odd) with the corresponding diagram as shown in Figure 2. There, β is the 3-braid either $S_2^{2b_1}S_1^{-2b_2}\cdots S_1^{-2b_m}$ or $S_2^{2b_1}S_1^{-2b_2}\cdots S_2^{2b_m}$ depending on whether m is even or odd. Any 2-bridge knot or link can be put in this form.

Let $P(b_1, b_2, \ldots, b_m)$, $V(b_1, b_2, \ldots, b_m)$, $\nabla(b_1, b_2, \ldots, b_m)$, $\Lambda(b_1, b_2, \ldots, b_m)$, and $F(b_1, b_2, \ldots, b_m)$ be the homfly, Jones, Conway, L, and Kauffman polynomials of $D(b_1, b_2, \ldots, b_m)$, respectively.

Proposition 1 ([18, Proposition 14]). *There holds*

$$P(b_1, b_2, \ldots, b_m) = (1, \mu)M(-b_1)M(b_2)\cdots M((-1)^m b_m)\begin{pmatrix}1\\0\end{pmatrix},$$

where

$$M(b) = \begin{pmatrix}(1-v^{2b})\mu^{-1} & 1\\ v^{2b} & 0\end{pmatrix}, \qquad \mu = (v^{-1}-v)z^{-1}.$$

From this proposition, we have

Proposition 2 ([18, p.128]). *There holds*

$$\nabla(b_1, b_2, \ldots, b_m) = (1, 0)N(-b_1)N(b_2)\cdots N((-1)^m b_m)\begin{pmatrix}1\\0\end{pmatrix},$$

where

$$N(b) = \begin{pmatrix}bz & 1\\ 1 & 0\end{pmatrix}.$$

Proposition 3 (cf. [26]). *Let* $\nabla_{-1} = 0$, $\nabla_0 = 1$, *and* $\nabla_m = \nabla(b_1, b_2, \ldots, b_m)$ *for* $m \geq 1$. *Then*

$$\nabla_m = (-1)^m b_m z \nabla_{m-1} + \nabla_{m-2}$$

for $m \geq 1$.

Therefore, if $b_i \neq 0$ for any i, then we have

$$\deg \nabla_m = m \tag{2}$$

for $m \geq 1$, and so the genus of $D(b_1, b_2, \ldots, b_m)$ is either $m/2$ or $(m-1)/2$ according as m is even or odd [4, 21].

Proposition 4 ([17, Theorem 5]). *There holds*

$$\Lambda(b_1, b_2, \dots, b_m) = (1, a^{-1}, d) S T^{2b_1-1} S T^{-2b_2-1} S \cdots S T^{(-1)^{m-1} 2b_m - 1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

where $d = (a + a^{-1}) z^{-1} - 1$,

$$S = \begin{pmatrix} 0 & 0 & a \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} z & 1 & 0 \\ -1 & 0 & 0 \\ z & 0 & a \end{pmatrix},$$

and so

$$F(b_1, b_2, \dots, b_m) = a^{-w} \Lambda(b_1, b_2, \dots, b_m),$$

where $w = 2(-b_1 + b_2 - \cdots + (-1)^m b_m)$.

3. Computational process

Step 1. Enumeration. We denote by $C(a_1, a_2, \dots, a_k)$ the unoriented 4-plat (or the unoriented diagram according to the context) as shown in Figure 3, where α is the 3-braid either $S_2^{a_1} S_1^{-a_1} \cdots S_1^{-a_k}$ or $S_2^{a_1} S_1^{-a_2} \cdots S_2^{a_k}$ according as k is even or odd.

An unoriented 2-bridge knot or link, or its mirror image, is uniquely represented as a 4-plat $C(a_1, a_2, \dots, a_k)$ satisfying the following conditions (3) and (4):

(3) $$a_1, a_k \geq 2, \ a_2, \dots, a_{k-1} \geq 1;$$

(4) either $a_i = a_{k-i+1}$ for all $i \geq 1$, or $a_1 = a_k$, $a_2 = a_{k-1}, \dots, a_{i-1} = a_{k+2-i}$, $a_i > a_{k+1-i}$ for some $i \geq 1$.

See [2, Proposition 12.13].

In order to enumerate all the 2-bridge knots and links of n crossings, we produce the sequences of integers $a_1 a_2 \cdots a_k$ satisfying (3),(4) and

(5) $$a_1 + a_2 + \cdots + a_k = n.$$

See [14, 22, 27].

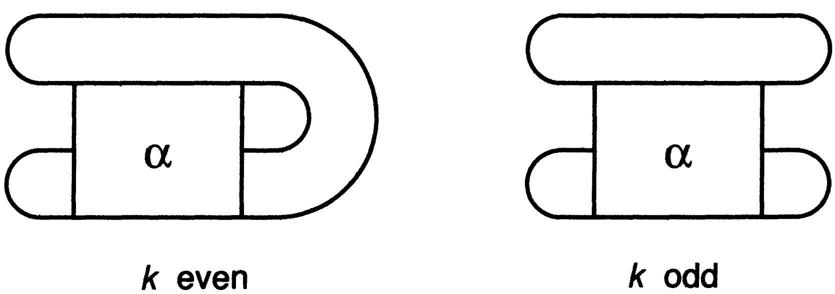

FIGURE 3

Specifically, we construct a binary tree as follows:

$$\begin{array}{ccccccc}
 & & & 2 & & & \\
 & & \swarrow & & \searrow & & \\
 & 3 & & & & 21 & \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
4 & & 31 & & 22 & & 211 \\
 & & & \vdots & & & \\
 & & & a_1a_2\cdots a_k & & & \\
 & & \swarrow & & \searrow & & \\
 & a_1a_2\cdots(a_k+1) & & & & a_1a_2\cdots a_k1 & \\
 \swarrow & & \searrow & & \swarrow & & \searrow \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}$$

Then choose the sequences of integers satisfying (3)–(5) from the $(n-1)$st row.

Calculate the coprime positive integers p and q by the continued fraction

$$\frac{p}{q} = a_1 + \frac{1}{a_2+} \cdots \frac{1}{+a_k}. \tag{6}$$

The 2-fold covering space of S^3 branched over $C(a_1, a_2, \ldots, a_k)$ is the lens space $L(p, q)$ [3]. See also [24, p. 303]. $C(a_1, a_2, \ldots, a_k)$ is a 2-bridge knot if and only if p is odd; p is the determinant of the 2-bridge knot $C(a_1, a_2, \ldots, a_k)$.

Take out the 2-bridge knots from the $C(a_1, a_2, \ldots, a_k)$'s and order them as follows:

$$C(a_1, a_2, \ldots, a_k) \prec C(a'_1, a'_2, \ldots, a'_l)$$

if either $p < p'$ or $p = p'$ and $a_1 = a'_1$, $a_2 = a'_2$, $\ldots$, $a_{i-1} = a'_{i-1}$, $a_i > a'_i$ for some i, where p and p' are the determinants of $C(a_1, a_2, \ldots, a_k)$ and $C(a'_1, a'_2, \ldots, a'_l)$, respectively.

Since a 2-bridge knot is invertible (cf. [2, Proposition 12.5]), we are not concerned with the question of knot orientation.

Let $\mathscr{K}_n$ denote the ordered set of the 2-bridge knots $C(a_1, a_2, \ldots, a_k)$ satisfying the conditions (3)–(5). Let

$$\overline{\mathscr{K}_n} = \{\, C(-a_1, -a_2, \ldots, -a_k) \mid C(a_1, a_2, \ldots, a_k) \in \mathscr{K}_n \}.$$

Then the union $\mathscr{K}_n^* = \mathscr{K}_n \cup \overline{\mathscr{K}_n}$ is the set of all the 2-bridge knots of n crossings, and the intersection $\mathscr{A}K_n = \mathscr{K}_n \cap \overline{\mathscr{K}_n}$ is the set of all the amphichiral 2-bridge knots of n crossings. It is known [26] that $C(a_1, a_2, \ldots, a_k)$ with (3)–(5) is amphichiral if and only if k is even and $a_i = a_{k+1-i}$ for all i. The numbers of these sets are explicitly given in [5].

Step 2. Calculation of the polynomial invariants. Let $K = C(a_1, a_2, \ldots, a_k) \in \mathscr{K}_n^*$ and p, q be obtained from (6). If q is odd (resp. even), then let $(r, s) = (q-p, q)$ (resp. $(q, q-p)$). Then K is the 2-bridge knot with Schubert's normal form $S(p, s)$. The classification theorem states that $S(p_1, q_1)$ and $S(p_2, q_2)$ are isotopic if and only if $p_1 = p_2$, $q_1^{\pm 1} \equiv q_2 \pmod{p_1}$. Also,

K is isotopic to $D(b_1, b_2, \ldots, b_m)$, where the b_i are obtained from the continued fraction

$$\frac{p}{r} = 2b_1 + \frac{1}{2b_2 + \cdots +} \frac{1}{2b_m}.$$

Compute $P(b_1, b_2, \ldots, b_m)$ and $F(b_1, b_2, \ldots, b_m)$ using Propositions 1 and 4. Next compute $\nabla(b_1, b_2, \ldots, b_m)$, $V(b_1, b_2, \ldots, b_m)$, and $Q(b_1, b_2, \ldots, b_m)$ by the substitutions as in the introduction. Note that $P_{\overline{K}}(v, z) = P_K(v^{-1}, z)$, $F_{\overline{K}}(a, z) = F_K(a^{-1}, z)$, $\nabla_{\overline{K}}(z) = \nabla_K(z)$, $V_{\overline{K}}(t) = V_K(t^{-1})$, and $Q_{\overline{K}}(z) = Q_K(z)$, where $\overline{K} = C(-a_1, -a_2, \ldots, -a_k) \in \mathscr{K}_n$.

Step 3. Comparison of the polynomial invariants. We have searched for all pairs of 2-bridge knots through 22 crossings having the same polynomial invariant. We first considered the Q polynomial. Let K be as in Step 2. Since the crossing number n of K equals the degree of $Q_K(z)$ plus one [15, 19] and $Q_K(2) = p^2$ [1], we sought pairs having the same Q polynomial in the set $\mathscr{K}_{n,p} = \{K \in \mathscr{K}_n \mid \text{the determinant of } K \text{ is } p\}$ for each n and p. Let K_1 and K_2 be such a pair in $\mathscr{K}_{n,p}$. We sought pairs having the same Jones polynomial in K_1, K_2, $\overline{K_1}$, $\overline{K_2}$. To do so, we compared the four pairs: $\{V_{K_1}(t), V_{K_2}(t)\}$, $\{V_{K_1}(t), V_{K_2}(t^{-1})\}$, $\{V_{K_1}(t), V_{K_1}(t^{-1})\}$, $\{V_{K_2}(t), V_{K_2}(t^{-1})\}$. If K_i, $i = 1, 2$, is amphichiral, we did not compare $\{V_{K_i}(t), V_{K_i}(t^{-1})\}$ and $\{V_{K_1}(t), V_{K_2}(t^{-1})\}$. When we found an equal pair, we examined their Kauffman and homfly polynomials. In addition, we compared $\{\nabla_{K_1}(z), \nabla_{K_2}(z)\}$ if K_1 and K_2 had the same genus.

4. Computational results

Combining Facts 1 and 2 in the introduction and Table 1, we know all the pairs sharing the same polynomial invariants.

In Table 1 in the Supplement section at the end of this issue, the three numbers "$p\ \ q, r$" represent the pair of the 2-bridge knots $\{S(p, q), S(p, r)\}$ in Schubert's notation. If there is no mark, $\{S(p, \pm q), S(p, \pm r)\}$ share the same Q polynomial. If there is a mark "V" (resp. "P", "F", "PF"), the pair $\{S(p, q), S(p, r)\}$ shares the Jones (resp. homfly, Kauffman, homfly and Kauffman) polynomial. If $S(p, q)$ is not amphichiral, $\{S(p, -q), S(p, -r)\}$ is also such a pair. If there is a mark "C", this pair shares the same Conway and Q polynomials. Note that we do not list the pair sharing only the same Conway polynomial. The mark "a" indicates that the knots are amphichiral. In this table, we have redundant information, for example, in 16 crossing knots, there are three pairs having the same Q polynomials: $\{S(429, 89), S(429, -353)\}$, $\{S(429, 89), S(429, -331)\}$, $\{S(429, -353), S(429, -331)\}$, which means the triple $\{S(429, 89), S(429, -353), S(429, -331)\}$ has the same Q polynomial. More complicated situations occur: Let $K_1 = S(1925, 569)$, $K_2 = S(1925, -1081)$, $K_3 = S(1925, 1229)$, which are 18 crossings. Then $P_{K_1} = P_{K_3}$ and $F_{K_2} = F_{K_3}$, and so $V_{K_1} = V_{K_2} = V_{K_3}$, $Q_{K_1} = Q_{K_2} = Q_{K_3}$, and $\nabla_{K_1} = \nabla_{K_3}$. Other equalities do not hold among them. Since they are not amphichiral, for the mirror images $\overline{K_i}$, $i = 1, 2, 3$, similar equalities hold.

In Table 2, for the n-crossing 2-bridge knots, we list the numbers of $\mathscr{K}_n$, $\mathscr{K}_n^*$ and the pairs listed in Table 1. From this table, we obtain Figure 4, which presents what proportion of the 2-bridge knots fail to be determined by the homfly and Kauffman polynomials.

TABLE 2

n	# K_n	# K_n^*	No mark	V	P	F	PF	C	Total
10	45	85	2	1	0	0	0	0	3
11	91	182	0	0	0	0	0	1	1
12	176	341	7	3	0	0	0	0	10
13	352	704	2	4	0	2	0	6	14
14	693	1365	14	18	4	0	1	0	37
15	1387	2774	17	23	1	4	0	7	52
16	2752	5461	77	46	20	1	2	0	146
17	5504	11008	65	73	5	10	0	27	180
18	10965	21845	202	161	40	5	7	2	417
19	21931	43862	229	244	13	22	0	72	580
20	43776	87381	593	498	82	14	17	17	1221
21	87552	175104	669	960	24	46	0	186	1885
22	174933	349525	1607	1751	236	46	31	47	3718

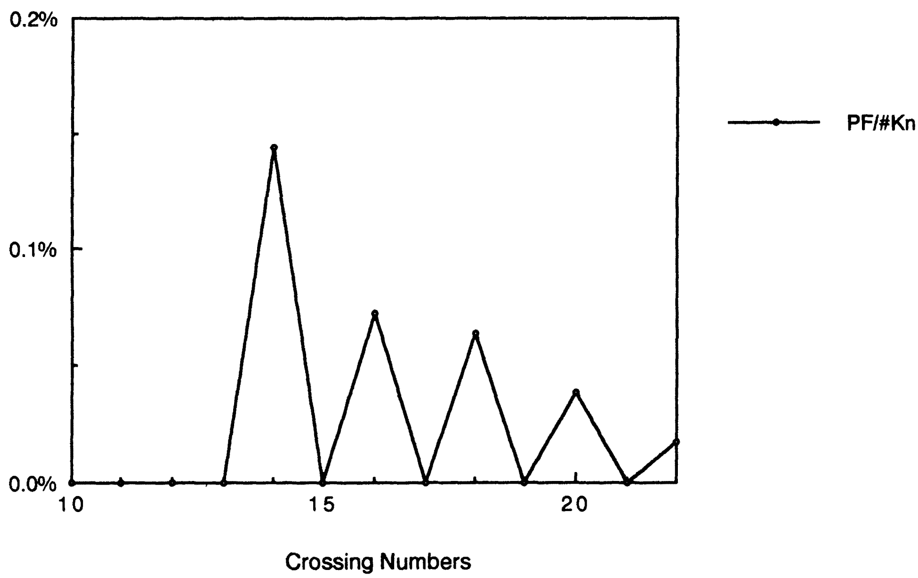

FIGURE 4

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